HOMEWORK 4

Problem 1 Let $\mu(n)$ be the Möbius function: $\mu(n) = 0$ if n is divisible by a square of an integer, and $\mu(n) = (-1)^k$, where k is the number of prime factors of a square-free number n. Prove that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \operatorname{Re} s > 1.$$

Problem 2 Let $\Lambda(n) = \log p$ if $n = p^\alpha$, where p is a prime and $\alpha \in \mathbb{N}$, and $\Lambda(n) = 0$ otherwise (von Mangoldt function). Prove that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \operatorname{Re} s > 1.$$

(Here $\zeta'(s)$ stands for the derivative).

Problem 3 Show that for $\operatorname{Re} s > 2$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s},$$

where $\varphi(n)$ is Euler's function.

Problem 4 Show that for $\operatorname{Re} s > 1$

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

where

$$\tau(n) = \sum_{d|n} 1$$

— total number of all positive divisors of the number n, including 1 and n.

Problem 5 Show that for $\operatorname{Re} s > 1$

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{e^{-x} x^{s-1}}{1 - e^{-x}} dx,$$

where $\Gamma(s)$ is Euler's gamma-function, defined for $\operatorname{Re} s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$